BIRATIONAL GEOMETRY EXERCISES

Exercise 0. Let $X = \mathbb{P}^n \times \mathbb{P}^m$. Define $\mathcal{O}(a,b) = p^*\mathcal{O}(a) \otimes q^*\mathcal{O}(b)$ where p and q are the projections onto the first and second coordinates, respectively. For what values of a and b is $\mathcal{O}(a,b)$... (1) ample, (2) big, (3) nef, (4) pseudo-effective?

Exercise 1. Find an example of (1) a nef divisor which is not ample, (2) a big divisor which is not nef, (3) a pseudo-effective divisor which is not big.

Exercise 2. Let X be a normal projective variety, let A be an ample line bundle and let L be a nef line bundle. Then $A \otimes L$ is an ample line bundle. Deduce that the nef cone is the closure of the ample cone.

Exercise 3. Let X be a normal projective variety and let A be an ample divisor. Show that $A \cdot C > 0$ for any projective curve $C \subset X$.

Exercise 4. Let X be a normal projective variety. Show that $\overline{NE}(X)$ does not contain a line.

Exercise 5. Let X be a normal projective variety and let D be a big divisor. Show that $H^0(X, \mathcal{O}(mD)) \geq Cm^{\dim X}$ for some $C > 0$.

Exercise 6. Let C be a smooth projective curve defined over an algebraically closed field. Show that if ω_C^* is ample then $C \cong \mathbb{P}^1$.

Exercise 7. Let X be a smooth surface such that K_X is pseudo-effective. Suppose that $K_X \cdot C < 0$ where C is a smooth curve. Show that $C \cong \mathbb{P}^1$. Find an example of a smooth surface X, a pseudo-effective divisor D and a curve C of genus $g > 0$ such that $D \cdot C < 0$.

Exercise 8. Let $n \geq 1$. Show that $X = \{xy + z^2 + w^{2n} = 0\} \subset \mathbb{A}^4$ is not $\mathbb{Q}$-factorial, i.e., there exists an effective divisor $D \subset X$ which is not $\mathbb{Q}$-Cartier.

Exercise 9. Let X be a normal projective variety and let A be an ample line bundle. Show that $A|_Z$ is ample for any subvariety $Z \subset X$. If A is big, is it the case that $A|_Z$ is big for any subvariety $Z \subset X$?

Exercise 10. Find an example of a flip (hint: it might be easier to find an example of a fourfold flip). Find an example of a flop which is not the Atiyah flop.

Exercise 11. Let X be the blow up $\mathbb{P}^3$ at 4 general points. Let $L_1, \ldots, L_6$ be the strict transform of the lines between any two points.

Show that L_i can be flopped. Show that this flop is locally isomorphic to the Atiyah flop.

Let $X \to W$ be the morphism which contracts all six lines and let $X \dashrightarrow X'/W$ be the rational map given by flopping all six lines. Show that $X' \cong X$ as varieties over $\operatorname{Spec} k$, but are not isomorphic as varieties over W.

Exercise 12. Let $F \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona involution $[x : y : z] \mapsto [x^{-1} : y^{-1} : z^{-1}]$. Show that F can be realised as the blow up of $\mathbb{P}^2$ at 3 points, followed by the blow down of three curves. Provide a similar description of the Cremona involuation on $\mathbb{P}^3$.

Exercise 13. Compute the minimal resolution of $\{xy - z^m = 0\} \subset \mathbb{A}^3$. Find a resolution of singularities of $\operatorname{Spec} k[x^2, y^2, z^2, xy, xz, yz]$. Can you find one with only one exceptional divisor?